

The Hawking temperature of expanding cosmological black holes

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In the context of a debate on the correct expression of the Hawking temperature of a cosmological black hole, we show that the correct expression in terms of the Hawking-Hayward quasi-local energy m_H of the hole is $T = (8\pi m_H(t))^{-1}$. This expression holds for comoving black holes and agrees with a recent proposal by Saida, Harada, and Maeda.

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INTRODUCTION

Recently, the power of Hawking radiation emitted by a cosmological black hole was computed by Saida, Harada, and Maeda [1]. In this work, the black hole is not the usual asymptotically flat spacetime but lives instead in an asymptotically Friedmann-Lemaître-Robertson-Walker (FLRW) universe, which is taken to have flat spatial sections for simplicity. In Ref. [1], two exact solutions of the Einstein equations describing such systems are considered: the Einstein-Straus vacuole [2] with a central non-expanding black hole, and the recent Sultana-Dyer solution [3] describing a perfectly comoving black hole embedded in a dust-dominated FLRW universe (see Refs. [4] for other dynamical black hole solutions and [5, 6] for the thermodynamics of dynamical black hole horizons).

The analysis of Saida, Harada, and Maeda [1] delivers two main results: for the Einstein-Straus black hole [2], which is not accreting, thermal radiation of quantum particles is suppressed by a factor coming from the expansion of the boundary between the local (static) black hole exterior and the expanding FLRW universe. This phenomenon is interpreted in analogy with radiation from an accelerated mirror, and we will not be concerned with it here. The second result, upon which we focus, pertains to the second exact solution studied in [1], *i.e.*, the Sultana-Dyer cosmological black hole [3]. This solution is obtained by conformally transforming the Schwarzschild metric

$$ds_{Schw}^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) = g_{ab}^{(Schw)} dx^a dx^b \quad (1)$$

according to

$$g_{ab}^{(Schw)} \longrightarrow g_{ab}^{(SD)} = \Omega^2 g_{ab}^{(Schw)} \quad (2)$$

and choosing the conformal factor $\Omega = a(t)$, the scale factor of a spatially flat FLRW metric

$$ds_{FLRW}^2 = -dt^2 + a^2(t) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] \quad (3)$$

with the particular choice $a(t) = a_0 t^{2/3}$ (the scale factor of a dust-dominated universe). The explicit goal of Sultana and Dyer is to turn the Schwarzschild global time-like Killing field ξ^c into a conformal Killing field (which happens for $\xi^c \nabla_c \Omega \neq 0$) which generates a conformal Killing horizon (the dynamical black hole horizon). The Sultana-Dyer metric can be written in various coordinate systems; for example,

$$ds_{SD}^2 = a^2(\eta) [-d\eta^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) + \frac{2m}{r} (d\eta + dr)^2] , \quad (4)$$

where $m = \text{const.} > 0$, η is the conformal time, and r is an areal (Schwarzschild-like) radial coordinate. To see that (4) is conformal to the Schwarzschild line element, one performs the coordinate transformation $\eta \equiv t + 2m \ln\left(\frac{r}{2m} - 1\right)$ which turns (4) into

$$ds_{SD}^2 = a^2(\eta) \left[-\left(1 - \frac{2m}{r}\right) d\eta^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] . \quad (5)$$

It is relevant [7] that the Sultana-Dyer solution can be seen as a generalization of the McVittie metric [8] describing a point particle embedded in a cosmological background. For a spatially flat FLRW background this is given by [8]

$$ds_{McVittie}^2 = -\frac{\left(1 - \frac{m(t)}{2\bar{r}}\right)^2}{\left(1 + \frac{m(t)}{2\bar{r}}\right)^2} dt^2 + a^2(t) \left(1 + \frac{m(t)}{2\bar{r}}\right)^4 \cdot [d\bar{r}^2 + \bar{r}^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] , \quad (6)$$

where \bar{r} is the isotropic radius defined by

$$r = \bar{r} \left(1 + \frac{m}{2\bar{r}}\right)^2 , \quad (7)$$

$a(t)$ is the scale factor of the background FLRW universe, and $m(t)$ is a function of the comoving time t related to

the physical mass of the central object and determined by the McVittie condition

$$\frac{\dot{m}}{m} + \frac{\dot{a}}{a} = 0, \quad (8)$$

where an overdot denotes differentiation with respect to the comoving time. This condition is equivalent to $G_0^1 = 0$ (where G_{cd} is the Einstein tensor) and to vanishing component T_0^1 of the energy-momentum tensor of the cosmic fluid. The physical meaning is that no accretion onto the central object occurs and, as a consequence, the central object's radius stays constant, as discussed in Ref. [7]. This dictates the form of $m(t) = \text{constant}/a(t)$. The McVittie metric (6) can not be interpreted as describing a cosmological black hole because at the putative horizon $\bar{r} = m/2$ (or $r = 2m$) the pressure of the cosmic fluid and the Ricci curvature R diverge [9, 10, 11]. An important exception is the Schwarzschild-de Sitter solution which is a special case of the metric (6) and describes a genuine static black hole embedded in de Sitter space [11].

The Sultana-Dyer solution corresponds to a generalized McVittie metric (6) in which the condition (8) is dropped and the central black hole is allowed to (radially) accrete the surrounding cosmic fluid. The Sultana-Dyer choice of $m(t) = \text{constant} \equiv m_0$ is, in this respect, the simplest. The Sultana-Dyer solution does not satisfy the energy conditions; indeed, the energy density of the cosmic fluid even becomes negative, and the flow becomes superluminal, near the horizon at late times [3]. Other solutions with similar problems have been presented in Refs. [12, 13], and new solutions have been found in Ref. [7]; some of the latter have positive energy density everywhere on the spacetime manifold but still suffer from the superluminal flow problem due to the oversimplified model of “rigid” accretion. However, they are not restricted to the special form $a(t) = a_0 t^{2/3}$ but hold for general scale factors.

The Hawking-Hayward [14, 15] quasi-local energy of the McVittie, Sultana-Dyer, and new solutions of the form (6) is [7, 11]

$$m_H(t) = a(t) m(t); \quad (9)$$

this should be regarded as the physical mass of the central object or black hole, where applicable, as opposed to the function $m(t)$ (which has led to misleading or incorrect statements in the literature [12, 16]) or to the Misner-Sharp mass [1] which is not particularly illuminating. In terms of m_H , the McVittie condition (8) says that the Hawking mass stays constant ($\dot{m}_H = 0$, no accretion), while the Sultana-Dyer solution has $m_H(t) = m_0 a(t)$, *i.e.*, is “perfectly comoving”. This is not an abuse of terminology because the physical mass m_H is related to the physical size of the horizon by $r = 2m_H = a(t)m_0$, which is reminiscent of the expression of the Schwarzschild radius $r = 2m$. This relation comes from the definition of

the areal radius $r \equiv \sqrt{\frac{\mathcal{A}}{4\pi}}$, where \mathcal{A} is the proper area of the horizon Σ ,

$$\mathcal{A} = \int \int d\theta d\varphi \sqrt{g_\Sigma} = 16\pi a^2 m^2 = 16\pi m_H^2 \quad (10)$$

and g_Σ is the determinant of the restriction $g_{ab}^{(\Sigma)}$ of the metric to this surface. Eq. (10) tells us that the surface $\bar{r} = m/2$ in the McVittie geometry (including the case of the Schwarzschild-de Sitter black hole) does not expand, while the Sultana-Dyer solution and the solutions of Ref. [7] are perfectly comoving.

THERMODYNAMICS OF A CONFORMALLY SCHWARZSCHILD COSMOLOGICAL BLACK HOLE

Let us consider now the zeroth law of black hole thermodynamics (*i.e.*, the surface gravity κ is constant on the horizon) for these cosmological black holes. In [3], Sultana and Dyer assumed the temperature of their black hole solution to be

$$T_{SD} = \frac{1}{2\pi} [\kappa_{DH} - \mathcal{L}_\xi (\ln \Omega^2)] = \frac{1}{8\pi m_0}, \quad (11)$$

i.e., constant over the conformal Killing horizon and equal to the temperature of the static Schwarzschild black hole conformal to the Sultana-Dyer solution. Here \mathcal{L} denotes the Lie derivative and ξ^c is the conformal Killing vector which becomes null on the conformal Killing horizon and satisfies

$$\mathcal{L}_\xi g_{cd} = (\mathcal{L}_\xi \ln \Omega^2) g_{cd}, \quad (12)$$

while the surface gravity of the dynamical horizon is defined by

$$\xi^c \nabla_c \xi^a = -\kappa_{DH} \xi^a. \quad (13)$$

Jacobson and Kang [17], instead, defined a generalized surface gravity κ_{JK} defined by the normalization of the conformal Killing vector as

$$\nabla_a (\xi^c \xi_c) = -2\kappa_{JK} \xi_a. \quad (14)$$

This is conformally invariant if $\Omega \rightarrow 1$ at infinity. The relation between these two notions of surface gravity is [1, 17]

$$\kappa_{JK} = \kappa_{DH} - \mathcal{L}_\xi (\ln \Omega^2), \quad (15)$$

from which it follows that the corresponding temperatures for the cosmological black hole coincide,

$$T_{SD} = T_{JK} \equiv T_{JKSD} = \frac{1}{8\pi m_0}. \quad (16)$$

However, this prescription for the black hole temperature is at odds with generalizations of the zeroth law to conformal Killing horizons existing in metrics that are conformal to asymptotically flat black holes [18, 19], and also with a simple argument proposed below. Saida, Harada, and Maeda argue that the black hole temperature should be the one given by the spectrum of the emitted Hawking radiation, which is instead [1]

$$T_{SHM} = \frac{T_{JKSD}}{\Omega} = \frac{1}{8\pi m_0 a} . \quad (17)$$

In the light of the previous discussion, this is simply $T_{SHM} = 1/(8\pi m_H)$, which appears natural when m_H is regarded as the physical mass of the Sultana-Dyer black hole and reduces to the usual $T^{(Schw)} = (8\pi m)^{-1}$ for a Schwarzschild black hole. This conclusion is supported by the following scaling argument. As discussed in great detail by Dicke [20] following earlier ideas of Weyl [21], a conformal transformation $g_{ab} \rightarrow \Omega^2 g_{ab}$ can be interpreted as a mere rescaling of the lengths of vectors and of the units used in a measurement, with the amount of rescaling depending on the spacetime position (although Dicke was concerned with the then-new Brans-Dicke theory [22], his argument is quite general and applies also to general relativity as well as other metric theories of gravity). All that is measured in an experiment is the *ratio* between a quantity q and its unit q_u . For example, the proper length of a ruler divided by the unit of length l_u is the same in the Minkowski metric η_{ab} and in a conformally related metric $g_{ab} = \Omega^2 \eta_{ab}$ if a new length unit $\tilde{l}_u = \Omega l_u$ is associated to it—see Ref. [23] for an application. Therefore, two metrics g_{ab} and \tilde{g}_{ab} are physically equivalent [32] provided that the units of the fundamental quantities length, time, and energy scale according to $\tilde{l}_u = \Omega l_u$, $\tilde{t}_u = \Omega t_u$, and $\tilde{m}_u = \Omega^{-1} m_u$ [20] (derived units are scaled accordingly to their dimensions). In this sense, there is no difference between using the Schwarzschild metric $g_{ab}^{(Schw)}$ and its conformal Sultana-Dyer cousin $\tilde{g}_{ab} = g_{ab}^{(SD)}$, provided that the units \tilde{l}_u , \tilde{t}_u , and \tilde{m}_u are appropriately scaled, *i.e.*, expanding for lengths and times, and redshifting away for energies. Since the black hole temperature T multiplied by the Boltzmann constant k_B scales as an energy, the ratio between $k_B T$ and m_u must be the same when using $g_{ab}^{(Schw)}$ or $g_{ab}^{(SD)}$, or

$$\frac{k_B \tilde{T}}{\tilde{m}_u} = \frac{k_B T^{(Schw)}}{m_u} , \quad (18)$$

which yields the effective temperature of the cosmological black hole

$$\tilde{T} = \frac{T^{(Schw)}}{\Omega} = \frac{1}{8\pi m_0 a} = \frac{1}{8\pi m_H} \quad (19)$$

in agreement with Ref. [1]. This simple argument supports the result of Saida, Harada, and Maeda [1]

and is fully consistent with the revealing use of the Hawking-Hayward quasi-local energy m_H rather than other mass notions. The argument does not apply to a Schwarzschild-de Sitter (or Einstein-Straus) black hole, which can not be obtained by conformally transforming the Schwarzschild metric (remember that $m_H = \text{const.}$ for this case, contrary to $m_H(t) = a(t) m_0$ for the Sultana-Dyer black hole).

In scalar-tensor cosmology it is well-known that simple rescaling provides the transformation law of the matter energy-momentum tensor under conformal transformations $g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}$ as

$$\tilde{T}_{ab}^{(m)} = \Omega^{-2} T_{ab}^{(m)} , \quad (20)$$

which agrees with a direct calculation of $\tilde{T}_{ab}^{(m)}$ [25, 26]. By applying the rescaling to the semiclassical stress-energy tensor of a scalar field in the background of a Sultana-Dyer (or any other comoving) black hole in our general-relativistic situation, the renormalized $\langle \tilde{T}_{ab} \rangle$ should then be

$$\langle \tilde{T}_{ab} \rangle = \frac{\langle T_{ab} \rangle}{a^2} . \quad (21)$$

The explicit renormalization of T_{ab} by Saida, Harada, and Maeda [1] instead yields

$$\langle \tilde{T}_{ab} \rangle = \langle T_{ab}^{(SD)} \rangle = \frac{\langle T_{ab} \rangle}{a^2} - \frac{1}{2880\pi^2} (X_{ab} - Y_{ab}) , \quad (22)$$

where [33]

$$X_{ab} = 2\tilde{\nabla}_a \tilde{\nabla}_b \tilde{R} - 2\tilde{g}_{ab} \tilde{\square} \tilde{R} + \frac{\tilde{R}}{2} \tilde{g}_{ab} - 2\tilde{R} \tilde{R}_{ab} , \quad (23)$$

$$Y_{ab} = -\tilde{R}_a^c \tilde{R}_{bc} + \frac{2}{3} \tilde{R} \tilde{R}_{ab} + \frac{1}{2} \tilde{R}_{cd} \tilde{R}^{cd} \tilde{g}_{ab} - \frac{\tilde{R}}{2} \tilde{g}_{ab} . \quad (24)$$

The extra terms in eq. (22) are interpreted as due to quantum particle creation by the expanding background [1], which could not be predicted by using Dicke's classical argument. However, when the black hole horizon is much smaller than the cosmological horizon, these terms can be safely neglected and the rescaling argument agrees with the proper calculation of $\langle T_{ab}^{(SD)} \rangle$ in [1].

An independent argument supporting the temperature (19) of cosmological black holes versus the expression (16) is the following. It is instructive to consider the first law of black hole thermodynamics which, for a static Schwarzschild black hole of mass m takes the form $TdS = dm$. The expression of the Bekenstein-Hawking entropy $S = A/4$, where $A = 4\pi r^2$ is the horizon area, together with the expression $r = 2m$ for the horizon radius, yields the Hawking temperature $T^{(Schw)} = 1/(8\pi m)$. For a conformally expanding black hole of the Sultana-Dyer type or of the type in Ref. [7], the quasi-local

energy $m_H(t) = a(t)m(t)$ and proper horizon radius $r_p(t) = a(t)r$ (as well as proper area $\mathcal{A} = 4\pi r_p^2$ and proper volume $V = 4\pi r_p^3/3$) should be used. For these expanding horizons, the first law of black hole thermodynamics includes a work term PdV :

$$TdS = dm_H + PdV. \quad (25)$$

By identifying again the black hole entropy with $S = \mathcal{A}/4$ and using proper quantities, one obtains

$$8\pi T m_H dm_H = dm_H + 32\pi P m_H^2 dm_H. \quad (26)$$

In the adiabatic approximation in which the accretion rate is small, the black hole is in a state of quasi-equilibrium and the work term can be neglected yielding

$$T \simeq \frac{1}{8\pi m_H(t)} = \frac{T^{(Schw)}}{a}, \quad (27)$$

in agreement with our previous argument. The disagreement between the result of Ref. [1], with which our arguments agree, and the Jacobson-Kang-Sultana-Dyer temperature is discussed in Ref. [1]. Recurrent folklore supports the idea that the black hole temperature is conformally invariant: however, the conformal invariance found in Ref. [17] is valid upon the assumption that the conformal factor satisfies $\Omega \rightarrow 1$ and that the conformal Killing field has unit norm at null infinity. These assumptions are not satisfied by the Sultana-Dyer black hole [3], nor by the comoving black holes of Ref. [7]. The radiation spectrum can be computed by evaluating the Bogoliubov coefficients relating ingoing and outgoing modes of positive and negative frequencies. The latter are defined by familiar boundary conditions when the spacetime is Minkowskian at infinity. These boundary conditions are not preserved by a conformal transformation mapping an asymptotically flat spacetime into an asymptotically FLRW one (in this case the Bogoliubov coefficients are not expected to be conformally invariant). The temperature of these black holes in the adiabatic approximation appears in Ref. [1] as a result of a calculation of the renormalized energy-momentum tensor (a full semiclassical calculation including explicit Bogoliubov coefficients is not yet available).

From the physical point of view, it is clear that the temperature of an expanding black hole must be time-dependent while, if it were conformally invariant, it would be constant in time for a conformally-Schwarzschild black hole. In fact, T is inversely proportional to the physical mass; the latter *must* be related with the physical radius of the horizon (*e.g.*, by the expression of the Schwarzschild radius $r_s = 2m$ for a Schwarzschild black hole). Therefore, since the horizon radius changes with time, also the physical mass changes with time, and so does the Hawking temperature. It would be unphysical for the temperature to remain time-independent while the black hole expands without bound.

STATIC CONFORMAL TRANSFORMATION

We now want to address an apparent contradiction [34] between the scaling argument proposed here and the claims of conformal invariance of the Hawking temperature appearing in the literature [3, 17]. While this contradiction does not exist for the cosmological black hole considered so far because conformal invariance of the surface gravity and Hawking temperature has been demonstrated only for scale factors that approach unity at spatial infinity, it is certainly legitimate to consider a *stationary* conformal transformation in which the conformal factor does not depend on time and approaches unity at infinity. One can then consider the conformally transformed Schwarzschild black hole with, say, $\Omega = \Omega(r)$ in order to preserve spherical symmetry, and $\Omega \rightarrow 1$ as $r \rightarrow +\infty$. The scaling argument still yields $\tilde{T} = \Omega^{-1}T$, in contradiction with the claim of conformal invariance $\tilde{T} = T$ [3, 17]. This contradiction disappears when one realizes that two different notions of temperature are used, and that quasi-local energy and quasi-local mass behave differently under conformal transformations. In the following we adopt the Brown-York notions of quasi-local energy and mass [27]. First, note that a stationary conformal transformation satisfies $\chi^c \nabla_c \Omega = 0$, where χ^c is the timelike Killing vector of Schwarzschild spacetime, and therefore the Schwarzschild Killing horizon is mapped into another Killing horizon, not just a conformal Killing horizon. Second, the Brown-York expression for the quasi-local mass is conformally invariant [28]: under general conformal transformations the latter is not a conserved charge, but it does enjoy this property for transformations with $\chi^c \nabla_c \Omega = 0$ [28]. However, it is not the quasi-local mass that should be used here but rather the (Brown-York) quasi-local energy which differs from the quasi-local mass and has been used extensively in quasi-local black hole thermodynamics [30]. The boundaries that are necessary to define quasi-local quantities, in general, may not be mapped into boundaries embeddable in the conformally related spacetime; however, this property holds for static, spherically symmetric, conformal transformations [29].

The quasi-local energy E is not conformally invariant but scales as $\tilde{E} = \Omega^{-1}E$ [28]. It is significant that, in the original Brown-York paper [27], the first law of thermodynamics applied to a Schwarzschild black hole with radius R and mass M becomes (eq. (6.20) of Ref. [27])

$$\frac{dS}{8\pi M \sqrt{1 - \frac{2M}{R}}} = dU + PdV, \quad (28)$$

where $S = A/4 = 4\pi M^2$ is the entropy. The equilibrium temperature here is given by $\left(8\pi M \sqrt{|g_{00}|}\right)^{-1}$, not simply by $(8\pi M)^{-1}$. This is reminiscent of Tolman's criterion for thermal equilibrium $T\sqrt{|g_{00}|} = \text{const.}$ [31]

which, applied to a Sultana-Dyer black hole, yields again $\tilde{T} = T/a$. For a stationary conformal transformation with $\Omega = \Omega(r)$, instead, if M is conformally invariant, the new temperature will be

$$\tilde{T} = \frac{1}{8\pi M \sqrt{|\tilde{g}_{00}|}} = \frac{T}{\Omega}, \quad (29)$$

which is consistent with the scaling relations

$$d\tilde{U} = \frac{dU}{\Omega}, \quad (30)$$

$$\tilde{P} = \Omega^{-4} P, \quad (31)$$

$$d\tilde{V} = \Omega^3 dV. \quad (32)$$

Eq. (31) is well known from the conformal transformation properties of perfect fluids in cosmology (*e.g.*, Ref. [26]), while eq. (32) applies to static conformal transformations for which $d\Omega = \dot{\Omega} dt = 0$ in any thermodynamic process. Therefore, the first law is valid also in the conformally rescaled world and a necessary condition for this to happen is that $\tilde{T} = \Omega^{-1} T$. The contradiction between the scaling argument and the claimed conformal invariance originates from two different definitions of temperature, one based on the quasi-local energy [27], and the other based on the conformally invariant surface gravity $\kappa_{JK} = \kappa_{DH}$ given by eq. (14) [3, 17] and, in this respect, akin to the Brown-York quasi-local mass

OUTLOOKS

The previous considerations re-open the issue of which notion of temperature is to be used as the physical temperature of a black hole that is conformally related to a static or stationary one or, more in general, of a dynamical horizon. We do not claim to have exhausted this subject here: this issue is still open and awaits clarification.

To conclude, we have given independent arguments supporting the result of [1] for the temperature of a Sultana-Dyer black hole. A simple interpretation of this temperature is given, which appears particularly natural once the Hawking-Hayward quasi-local energy m_H is adopted as the physical mass. The prescription (19) for the temperature of comoving cosmological black holes is extended to the solutions of Ref. [7].

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- [32] In modern language and in generalized gravity theories,

this is known as the equivalence between the *Jordan conformal frame* and the *Einstein conformal frame* (for a recent discussion see Ref. [24]).

[33] Beware of the fact that tilded and non-tilded quantities

are inverted in the notations of Ref. [1].

[34] We thank a referee for bringing up this point.